From: lsmactsai
Sent: Friday, May 13, 2022 9:18 AM
To: MATHFRAMEWORK
Subject: [EXTERNAL] comments on math framework 2nd field review drat

Dear Sir/Madam,
CDE

Comments shown below..

Martin Tsai

Overview. Latest revised framework is full of inadequately researched math education philosophies and pedagogy promoted that are not backed up by sound research evidence. As stated by Ze'ev Wurman and Williamson Evers "the framework's lack of any recommendations regarding the proven effective spaced (or distributed) practice — the use of homework and quizzes intentionally spread over a period of weeks after learning a topic, to maximize retention. The focus on inquiry learning, which relies heavily on students' struggles, has been discouraged by strong research." Curriculum needs to simultaneously develop conceptual understanding, computational fluency, and problem-solving skills and the framework is biased against this proven approach.

The authors of the framework should have consulted more with faculty in the STEM fields who are more familiar with the advanced mathematics education and training students need after high school. Additionally, inputs from and involvement with STEM professionals in the development of the framework, including aerospace industry experts and/or researchers, those who have first-hand knowledge of what mathematics training and background are needed for success in the STEM industry, are sorely missing.

In the technology cold war with China: US/CA need hard science professionals, engineers & CS at the fore-front of technology as soldiers. Yet we are shying away from that in the math framework by de-emphasizing calculus. Instead high school students are being steered to take "data science," with few published data or track record, as a viable alternative to advanced algebra and calculus courses that are designed to prepare them for college-level math courses. The described data science skills such as data cleaning, downloading/uploading while useful are no replacement for the mathematical foundations necessary to succeed in STEM programs in college and beyond. In Ontario, Canada, "data science" type classes is named "Math of data management" and is of interest for students planning to enter university programs in business, the social sciences, and the humanities, but not in STEM fields.

Not only will the well prepared students be negatively affected by the framework's restrictive policy towards calculus taking in high school, the impressive improvement of

Hispanic/Latinx students in the numbers who passed AP calculus that rose from barely over 1,050 in 1999 to 8,100 in 2019, roughly an 8-fold increase, see comment D below). Adoption of the framework will serve to discourage the proper recognition and cultivation of talented students and ultimately deprives california's vital aerospace and defense industry of a qualified diversified group of STEM graduates.

Nowadays, there are still far more students who pursue calculus classes than statistics at both the high school and college levels. The wide availability of computers means that we should focus more on mathematical reasoning, not less. Calculus remains as an admission requirement or highly recommended for competitive colleges in CA and US such as Pomona colleges, CalTech, Carnegie-Mellon, Cornell, and MIT. College admissions aside, students who have learned the material of AP calculus are at an advantage in STEM programs, since they can often place out and enjoy a lighter workload and/or increased opportunities.

As stated earlier, the framework, with its focus on inquiry learning which relies heavily on students' struggles. A review of much of the research cited in the framework reveals that what the framework describes as "clear" is often actually pretty murky, or contradicted by other research, e.g., that 'a 3 years pathway' is sufficient preparation for AP calculus, contradicting ICAS's position of 4 years required, etc. The framework's weak and/or frequent contradictory research foundation does not inspire confidence that its goal – improvement in preparing students for their post-secondary and career endeavors will ever be met. For these reasons, ***I am obligated to cast an unequivocal NO and turn*** thumbs down on the mathematics framework.

The revised framework field review draft appears to be rushed, not well organized, and rife with inconsistencies and contradictions as delineated below. Presented in this Overview Section are the issues and comments denoted from A through D, these are followed by 6 additional issues presented in the 'Additional Issues/Comments" Section to follow.

Issue A): Algebra I before high school. *Revised Math Framework, Chapter 8, pp. 32, lines 792-801.*

Currently, most high schools require courses in Algebra, Geometry, Algebra 2.... This sequence means that students cannot easily reach Calculus unless they have taken a high school algebra course in middle school. This has led to many students missing … middle school mathematics, often by skipping the grade eight course, or by taking compressed courses. Among the problems with this approach is that some students who take eighth grade Algebra … may miss foundational learning, and those who do not take that course are filtered out of the calculus pathway early on, with significant racial and gender inequalities (Joseph, Hailu, and Boston, 2017).

-----------(Begin comment A) --
Comment A). U.S. Department of Education (USDE, 2018) reported "We know that a strong STEM education is a path to successful career and that the need for STEM knowledge and skills will continue to grow in the future. Taking Algebra I before high

school, such as in 8th grade, can set students up for a strong foundation of STEM education and open the door for various college and career options."

(USDE, 2018) U.S. Department of Education. A leak in the STEM Pipeline: Taking Algebra Early. Nov. 2018, as given in https://www2.ed.gov/datastory/stem/algebra/index.html

Furthermore, RAND Corp. researcher McEachin et al. (2019) articulated "We find that enrolling in 8th grade Algebra boosts students' chances of taking advanced math courses in high school by 30 percentage points in 9th grade and 16 percentage points in 11th grade ... women, students of color, and English-Language Learners benefit disproportionately from access to accelerated coursework." He further quantified "enrolling in 8th grade Algebra substantially boosts students' advanced mathematics courses taking rates ...ultimately increasing students' likelihood of being on track to take Calculus in 12th grade by approximately 16 percentage points."

(McEachin et al. 2019) One course, many outcomes: A multi-site regression discontinuity analysis of early Algebra across California middle schools (EdWorkingPaper: 19-153), pp1 and pp. 25, 2019. https://www.edworkingpapers.com/sites/default/files/ai19-153.pdf

Regarding the article by *(Joseph, Hailu, and Boston, 2017)* in which the controversial critical race theory (CRT) is used to explore factors that contribute to Black women and girls' persistence in the mathematics pipeline and how these factors shape their academic outcomes, one fails to find any reference on "those who do not take that course are filtered out of the calculus pathway early on."

California's average NAEP scores in grade 8 math generally are below those for Massachusetts and Minnesota. These data indicate a root cause behind California's challenges in grade 8 mathematics, students' math preparation in earlier grades is inadequate and suffers beginning in grade 5. But instead of improving the students' math preparation in earlier grades, framework's philosophy and approach is to aim at the lowest common denominator, and in the name of equity, hold back students from advancing to algebra I in 8th grade, and claim that by so doing will in fact make students 'achieve more' which defies logic.

Suggested change: Let school districts, students and parents jointly decide whether it is advantageous for students to take an Algebra I course in middle school. This is to be accompanied by concurrently improving the students' math preparation in earlier grades starting in grade 5. Delete reference to the controversial research *(Joseph, Hailu, and Boston, 2017).*

------------(End comment A)---

Issue B): A 3-year pathway to 12th grade calculus. *Revised Math Framework, Chapter 8, pp. 32-33, lines 807-810, pp. 33, lines 813-816, and pp. 34, lines 854-857*

"..., it would be desirable to consider how students who do not accelerate in eighth grade can reach higher level courses, potentially including Calculus, by 808 twelfth grade. One possibility could involve reducing the repetition of content in high school, so that students do not need four courses before Calculus."

"…how high school course pathways may be redesigned to create a more streamlined three-year pathway to pre-calculus/calculus or statistics or data science, allowing students to take three years of middle school foundations and still reach advanced mathematics courses…"

"… indicates three possible pathways for high-school coursework, reflecting a common ninth- and tenth-grade experience, and a broader array of options in eleventh and twelfth grade. High schools will typically offer one of the first-two-years pathways (Integrated, MIC, or Traditional), and an array of more advanced courses."

-----------**(**Begin comment B) --

Comment B). Framework effectively claims with little evidence that 3 years of high school mathematics is sufficient foundation for students to reach calculus by 12th grade. This is a clear departure from the NCTM standards as stated by Inter-segmental Committee of the Academic Senates (ICAS) (2013) that "physical science and engineering majors should have successfully completed 4 years of secondary school level mathematics prior to taking calculus in high school." The mostly unproven 3-year pathway proposed by framework authors amounts to "an irresponsible compression late in high school" as proclaimed by STEM professional and college educators in the open letter (OPEN, 2021). The open letter added "it is irresponsible to make such radical (and detrimental) recommendations for the education of students in our largest state based on inconclusive or cherry-picked evidence." Incidentally the signatories of the open letter include several Nobel Prize winners, five Fields medalists and three Turing award winners, as well as more than 200 professors from the UC system, USC and Stanford universities. Their concerns should be addressed.

For students who do not take Algebra I in eighth grade but still wish to complete the 4 mathematics classes to reach higher level courses like calculus by 12th grade, families with means can find workarounds – such as private instructions and summer school – to ensure their children complete the extra course. There is, however, a much less burdensome and proven approach the school district can adopt which is to have compacted classes starting in 9th grade and continue through and including 11th grade. In other words, compacting 3 years of standard high school mathematics and 1 year of pre-calculus courses into 3 years of more intense mathematics is a lot less demanding than squeezing Algebra II and pre-calculus into a single compressed course late in high school. This compacted-classes approach for math has been successfully implemented in CA's Irvine school district and in MA. But this mis-guided framework for political and supposedly equity issues has so far refused to adopt this sensible approach (framework mandates all students to have common ninth- and tenth-grade experience), so students with the fewest resources will likely end up adversely impacted.

(ICAS, 2013) Inter-segmental Committee of the Academic Senates (ICAS). Statement on Competencies in Mathematics Expected of Entering College Students, April 2010, revised May 2013 as given in https://icas-ca.org/wp-content/uploads/2020/05/ICAS-Statement-Math-Competencies-2013.pdf

(OPEN, 2021) STEM professionals and college educators. Open Letter on K-12 Mathematics. 2021. https://sites.google.com/view/k12mathmatters/home?adobe_mc=TS%3D1638770283%7CMCMID%3D558304451

28708458578967747253879889721%7CMCORGID%3DCB68E4BA55144CAA0A4C98A5%40AdobeOrg&wsj_native_webview=androidphone

------------(End comment B)---

Issue C): Impact of taking a HS calculus course. *Revised Math Framework, Chapter 8, pp 41, lines 1044- 1047.*

Indeed, in a large national study across 133 institutions, Sadler and Sonnert (2018) found that mastery of the mathematics considered preparatory for calculus had, on average, more than double the positive impact of taking a high school calculus course on students' later performance in college calculus.

-----------(Begin comment C) ---

Comment C). The framework statement appears to be taken out of context as the authors are trying to equate mathematics considered "preparatory for calculus" with "pre-calculus" which is not valid. On pp. 57 of Sadler et al. (2016), the "preparation for calculus" variable is defined as a normalized composite constructed after a factor analysis that showed a strong relationship between high school grades in non-calculus mathematics (i.e., algebra I, geometry, algebra II, pre-calculus and students' SAT or ACT quantitative score). So mastery of the "preparation for calculus" means the student is strong in multiple areas of math. In the summary section of Sadler (2016), pp. 63-64 is stated "Among the students in the introductory college calculus classes, those who have taken high school calculus earn a grade half a letter higher, on average, compared with students with a similar pre-calculus preparation, but without a high school calculus course. The half of college calculus students who have taken calculus in high school generally appear to have gained a deeper understanding of algebra, geometry, and pre-calculus from their studies." So putting two and two together, the best preparation for college calculus is to do both: achieving mastery of algebra and geometry and pre-calculus, and taking calculus in high school.

In CA, most students who studied AP calculus in secondary schools (66,400 in 2019) and 47,700 of examinees ended up passing the AP calculus exam for an impressive 72 percent. Hargrove, Godin, and Dodd (2008) found that AP Calculus AB Examinees earned first-year GPAs that were significantly higher than non-AP students of similar ability and background. This finding held for fourth-year GPAs as well. According to Patterson, Packman, and Kobrin (2011), whose sample included students at 110 colleges and universities, AP mathematics examinees outperform non-AP students of similar ability and background in their mathematics subject GPA.

So the positive impacts on subsequent calculus performance in college for students who study calculus in high school and/or take AP calculus test are well documented.

As was discussed (see Comments A &B), in order to arrive at calculus by 12th grade, students can either take Algebra I in middle school or for those who defer it till 9th grade, school districts should offer and students can take a compacted "3 years of standard high school mathematics and 1 year of pre-calculus courses" into "3 years" sequence in grade 9th through 11th.

(Sadler, 2016) Sadler P. and Sonnert, G. The path to college calculus: the impact of high school coursework. In D. Bressoud (Ed.) The Role of Calculus in the Transition from High School to College Mathematics. Workshop held at the MAA Carriage House Washington, DC, pp 53-65, March 17–19, 2016.

Hargrove, L., Godin, D. and Dodd, B. (2008). College Outcomes Comparisons by AP® and Non-AP High School Experiences. New York: The College Board. Mattern, K.D., Shaw, E.J., and Xiong, X. (2009).

Patterson, B.F., Packman, S., and Kobrin, J.L. (2011). Advanced Placement® Exam-Taking and Performance: Relationships with First-Year Subject Area College Grades. New York: The College Board.

Suggested change: The cherry-picked research cited to support the narrative against taking calculus in high school is contradicted by other research which establishes the benefit of subsequent improved calculus grades in college by taking high school calculus.

------------(End comment C)---

Issue D): Race to Calculus. *Revised Math Framework, Chapter 8, pp 41-42, lines 1048- 1063.*

"The Mathematical Association of America (MAA) and NCTM issued a statement to urge that "the ultimate goal of the K–12 mathematics curriculum should not be to get into and through a course of calculus by twelfth grade, but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college" (Bressoud, 2012). The UC Board of Admissions and Relations with Schools (BOARS) made a similar statement: "

"...not to race to calculus Indeed, students whose math classes are at a mismatched level— either too advanced or too basic—often become frustrated and lose interest in the topic." (BOARS, 2016)

-----------(Begin comment D) ---

Comment D. Note that the (MAA) and NCTM statement was published around 2012, ten years have since elapsed and some of the background data Bressoud (2012) quoted to support the 'statement' is 30 years old, and decidedly obsolete. Specifically, Bressoud (2012) stated "the high school class of 1992, one-third (33.3%) of those who took calculus in high school then enrolled in pre-calculus when they got to college" is no longer applicable to the US/CA calculus landscape. As evidence of students' improved performance in calculus since 1992, Bressoud et al. (2015), pp. 9, Table 9, depicts that across the country in 2010 the percentage of students who took pre-calculus at a PhD granting university (e.g., UC) decreased to 13%, a reduction of 20 percent points from 1992, a downward trend likely to have continued till today as both the HS calculus curriculum and teacher development have improved considerably over the years.

(Bressoud, 2015) Bressoud, Mesa, and Rasmussen (Eds.) Insights and recommendations from the MAA national study of college calculus. MAA Press, 2015.
https://www.maa.org/sites/default/files/pdf/cspcc/InsightsandRecommendations.pdf

Additionally, due to the high growth in both the number of students taking and the number passing the AP-cal test (scored a 3 or higher), in CA in 1999 there were 16,500 students who passed AP-calc exam and their head count rose to 47,700 in 2019, an almost a factor of 3 increase. Figures 1 and 2 exhibit how the racial and ethnic distribution of the student who scores a 3 or higher on either AP calculus exam has changed over the past twenty years.

Figure 1. Distribution by race/ethnicity of California Test-Takers in 1999 who scored a 3 or higher on either AP calculus Exam. Source: College Board [9, 1999] and [17, 2019]

Figure 2. Distribution by race/ethnicity of California Test-Takers in 2019 who scored a 3 or higher on either AP calculus Exam. Source: College Board (1999), and (2019).

Although the percentage of White students who passed AP calculus exam dropped over the years, nonetheless the number of White students earning 3 or higher increased, almost two-fold to 13,400 in 2019 in proportional to their increased count of graduates.

Even though the percentage of Black students among high school graduates have dropped over two decades, it is encouraging that presence among those earning 3 or higher has remained at 1%, indicating an improved passing rate over time. For comparison, Asian students (including Filipino students) among high school graduates declined slightly over the years while their presence among students earning 3 or higher rose by 6.1 percent points.

Notably, Hispanic/Latinx students have shown a significantly increased presence among students earning 3 or higher as during this time their proportion surged from 6.4% to 17.2%, which is partly but not totally accounted for by their higher proportion among high school graduates. The improvement of Hispanic/Latinx students in numbers is even more impressive as barely over 1,050 of them passed in 1999 and the number rose to 8,100 in 2019, roughly an 8-fold increase.

A recent and relevant recommendation clarifying calculus's role in CA math education can be found in BOARS (2016) Statement on the Impact of Calculus on UC Admissions. In answer to the question of whether calculus should be offered in high schools, a reply of "Yes, definitely" was given. It further stated "Studying calculus in high school is especially helpful for students majoring in STEM fields, where coursework is highly sequential. … BOARS strongly recommend that high schools maintain calculus as an option for enthusiastic, well-prepared students."

Judging from the AP calculus test scores improvements over the past 2 decades coupled with significant decrease in percentage of students who took pre-calculus at a PhD granting university (e.g., UC), the advice not to 'rush through high school math without adequate understanding of fundamentals' has apparently been well heeded. For those who are prepared and inclined for a post-secondary experience in STEM fields, there is no justifiable reason for the framework to disrupt and impede their successful mathematics journey to calculus in high school that students have enjoyed for decades in the largest US state.

(CB, 1999) Advanced Placement Program, the College Board. 1999 AP State Summary Report Tables. California, All Candidates, School AP Grade Distributions by Total and Ethnic Group. https://secure-media.collegeboard.org/digitalServices/pdf/research/ca_1999.pdf

(CB, 2019) Advanced Placement Program, the College Board. California summary 2019. School AP Score Distributions by Total and Ethnic Group. https://research.collegeboard.org/programs/ap/data/archived/ap-2019

(BOARS, 2016) UC Board of Admissions and Relations with Schools (BOARS). Statement on the Impact of Calculus on UC Admissions UC, April 2016 as given in https://senate.universityofcalifornia.edu/_files/committees/boars/documents/ BOARS_Statement-Impact-Calculus.pdf

------------(End comment D)--

Other than the A through D issues and comments described above, next discussed are 6 additional issues and comments.

Additional Issues/Comments

They are: 1) on "far more students pursue statistics classes than calculus," 2) Rotations, 3) Population Growth function, 4) Model with mathematics, 5) Equation of a line, and 6) Recursive Sequences. These issues and comments (6 total) are presented below.

Issue 1): On "far more students pursue statistics classes than calculus…". *Revised Math Framework, Chapter 5, pp. 66-67, lines 1593-1596.*

With the rapid expansion of information available to all in the form of data, far more students pursue statistics classes than calculus, and may be better served by a data science course as a culminating high school mathematical science experience.

------------(Begin comment 2)--
Comment 1):. From CA Department of Education (CDE) database-2019, the last year data are available, a total of 66,400 middle/high school students* (rounded to the nearest hundredth) were enrolled in AP calculus courses compared with 38,600 students who were enrolled in AP statistics courses. For non-AP calculus courses, 44,600 students** enrolled in PreCalc and Math Analysis compared with 34,400 enrolled in probability and statistics. So CDE data indicate at the middle/high school level, more students took calculus classes than statistics at both AP and non-AP levels. Other academic years show similar patterns: more students took calculus than statistics.

* 47,300 and 19,000 students (rounded to nearest hundredth) enrolled in AP Calc AB and AP Calc BC, respectively.
** 23,900 and 20,700 students enrolled in PreCalc and Math Analysis, respectively.

Inspection of College Board Summary-2019 reveals that in CA, 70,500+ students took AP calculus compared with 33,000 who took AP statistics, indicating the importance placed by students (and colleges) on calculus over statistics.

+48,100 and 22,400 students took College Board AP Calc AB and AP Calc BC, respectively.

Then at the college level, as documented in US Department of Education NCES (2020) in 2018-2019, 126,700 and 86,700 bachelor's degrees were conferred to graduates in the engineering and information-and-computer science fields, respectively. With few exceptions, majors in these fields are required to take a sequence of calculus classes early in their degree program. In contrast, the mathematics and statistics fields' graduates garnered only 26,100 bachelor's degrees. The data clearly indicates in the real world there are far more students who pursue calculus than statistics classes and is diametrically opposite to what is claimed by framework authors.

(NCES, 2020) https://nces.ed.gov/programs/digest/d20/tables/dt20_322.10.asp

Suggested change: Retract the statement "*far more students pursue statistics classes than calculus*" as that is not borne out by facts/data.
------------(End comment 1)--

Issue 2): Rotations. *Revised Math Framework, Appendix A, pp 22-23, lines 232-246.*

Defining Rotations

Mrs. B wants to help her class understand the following definition of a rotation:A rotation about a point P through angle α is a transformation A⟼A^' such that (1) if point A is different from P, then PA=PA^' and the measure of ∠APA^'=α; and (2) if point A is the same as point P, then A^'=A."

...

"While justifying that the properties of the definition hold for the shapes she has given them, the students also make some observations about the effects of a rotation on the entire plane, for instance that:

> *Rotations preserve lengths.*
> *Rotations preserve angle measures.*
> *Rotations preserve parallelism."*

------------(Begin comment 2) ---

Comment 2): No need to introduce the property "Rotations preserve parallelism."

First, a polygon can be subdivided into triangles. From the first two properties stated "rotations preserve lengths," and "rotations preserve angle measures," then two triangles, before and after rotations are congruent. It follows then that polygons before and after rotations, are congruent since the underlying sub-divided triangles are. There does not seem to be any meaningful additional insight that one wishes to impart to students to introduce "preservation of parallelism" as a separate property of rotations. Besides, some objects do not have parallel sides to begin with, so although "rotations preserve parallelism" is satisfied vacuously in that case, one might run into the danger of misleading and confusing students to look for "non-existent" parallel lines in the original object.

<u>Suggested change</u>: Either i) delete the "Rotations preserve parallelism" property which is redundant, or ii) replace the property with "Rotations preserve congruence."

------------ (End comment 2) ---

Issue 3): Population Growth function. *Revised Math Framework, Appendix A, pp 26-28, lines 289- 329.*

"Example (Adapted from Illustrative Mathematics 2013)

Population Growth. The approximate United States Population measured each decade starting in 1790 up through 1940 can be modeled by the function *where* t *represents decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.*

Some possible questions:

a) According to this model for the U.S. population, what was the population in the year 1790?
b)….
c) …
d) For larger values of t, such as t= 50, what does this model predict for the U.S. population? Explain your findings.
…
…
Students see how the visual displays and summary statistics learned in earlier grade levels … and the role of randomness and careful design in the conclusions that can be drawn.

------------(Begin comment 3) **---**
<u>Comment 3):</u> Population growth function P(t) should be simplified.

The P(t) function should be simplified before attempting to obtain solutions to questions, a practice often advocated by framework authors, especially in this case as the function P(t) appears to be unnecessarily complicated with two "exponential" terms, one in the numerator and the other in the denominator. To simplify, first divide numerator and denominator by 200 million, and then divide both numerator and denominator by exp(0.31t). After combining the exp(-0.31t) terms yields

It is seen that PS(t) is characterized by a single "exponential" term with negative power involving "t" in the denominator as opposed to two "positive exponential" terms in P(t).

One can answer for question *a)* "the population in the year 1790 (when t = 0)" which is given by PS(0) and equals 3,900,000, since =1 and the denominator sums to 1.

To answer question d) "*what does this model predict for the U.S. population* for large values of t such as t= 50," then observing that approaches 'zero' as t is large such as t =50, then PS(t) reduces to a ratio of two constants 3,900,000 and 0.0195 = 200 million.

For intermediate values of t (decades *starting in 1790 up through 1940 and/or intervening years),* then PS(t) can be simply evaluated with the aid of an "exponential" function Table or a standard calculator, just as one would in calculating P(t) albeit with fewer (reduced number of) steps.

PS(t) can alternatively be scaled to the following equivalent form with unity coefficient for the exponential term:

which is completely characterized by 3 coefficients

a = 3,977,562, b = 0.019888, and c = 0.31.

When the 3 coefficients are all equal to 1, then the scaled PS(t) function becomes the well-known Sigmoid function used in logistic regression classification algorithms (to return a probability value).

Suggested change: Simply population growth function P(t) to either the PS(t) or scaled PS(t) forms, with a single exponential term in denominator.

------------**(**End comment 3) --

Issue 4): Model with mathematics. *Revised Math Framework, Appendix A, pp 42, portion of Figure A.11, embedded within lines 492-493.*

Figure A.11: Standards for Mathematical Practice—Explanation and Examples for Mathematics

SMP.4 ***Model with mathematics.***	*Students* ***apply their mathematical understanding of linear and exponential functions*** *to many real-world problems,* ***such as*** *linear and* ***exponential growth****. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.*

------------**(**Begin comment 4) --

Comment 4): HS students should attempt to derive PS(t) as an exercise, where PS(t) is the scaled form of the "*population growth function" P(t) as shown in framework*, *Appendix A, pp 26-28, lines 289- 329.* See also comment 3) above.

The population growth function P(t) as given in the framework is

As shown in comment 3) previously, P(t) after scaling becomes

which is completely characterized by 3 coefficients

a = 3,977,562, b = 0.019888, and c = 0.31

The coefficients "a, b and c" of the scaled PS(t) can be determined in a straight-forward manner by HS students, using a few population data points as inputs. US census population data for each decade from 1790 up through 1940, denoted by Pcens(t); for t = 0,1,2, ..., 15, can be collected by simple Google search. Pcens(t) for t=0, 2, 14 and 15, are explicitly given below

[Pcens(0), Pcens(2), Pcens(14), Pcens(15)]
= [392,9214, 723,9881, 122,775046, 132,164,569]

Solve the following 3 equations for the 3 unknown coefficients a(i), b(i) and c(i).

Step 1. For i = 1.

i) Solve for a(i) in terms of US census population Pcens(t) at t = 0
a(i) = Pcens(0)(1+b(i-1)) = Pcens(0)
= 3,929,214; where b(i-1) was initialized to 0, for i = 1 .

ii) Solve for c(i) in terms of US census populations at t = 0 and t =2, and the known coefficient b(i-1).

$$c(i) = -\ln(Pcens(0)/Pcens(2) - (1 - Pcens(0)/Pcens(2))*b(i-1))/2$$
$$= -\ln(0.54272)/2; \quad \text{since } b(i-1) = 0 \text{ for } i = 1.$$
$$= 0.30558; \quad .$$

after substituting Pcent(0)/Pcens(2) = 0.54272 into the c(i) expression.

iii) Solution for b(i) can be expressed in terms of US census populations at t = 14 and t =15, and the known a(i) and exp(-ct) terms.

$$b(i) = 0.5(a(i)/Pcens(15) - \exp(-15c(i)) + a(i)/Pcens(14) - \exp(-14c(i)))$$
$$= 0.5(3,929,214 (1/132,164,569+ 1/122,775,046)$$
$$- \exp(-4.58371) - \exp(- 4.27813)); \quad \text{since } a(i) = 3,929,214 \text{ for } i =1.$$
$$= 0.018824$$

Step 2. With a(i), b(i) and c(i) determined for i =1, repeat solving the 3 equations for improved coefficient estimates for i = 2, 3, …

After one iteration (up to i=2) for a total of 6 evaluations, the following coefficient estimates of the population function scaled PS(t) are obtained

a(2) = 4,003,200 ~= 3,978,000 (coefficient 'a' of scaled PS(t))
b(2) = 0.0207 ~= 0.02 (coefficient 'b' of scaled PS(t)), and
c(2) = 0.3136 ~= 0.31 (coefficient 'c' of scaled PS(t)).

The above numerically determined coefficients of the population function are practically identical to the coefficients of the scaled PS(t)**. The coefficients were obtained by iteratively solving 3 equations synthesized from population "input" data points (Pcens(0), Pcens(2), Pcens(14), and Psens(15)) for the 3 unknown coefficients, with some algebraic manipulations and requiring no more than the aid of a calculator.

** It can be shown that the populations computed using the numerically derived PS(t) and the scaled PS(t) function for decades from 1790 up through 1940 ("t" from 0 through 15) differ, in the "sum of root-mean-squared error" sense, by less than 1%.

Hands-on experience in deriving the scaled PS(t) function as population growth model builds confidence and demonstrates to HS students that they have the requisite repertoire of math tools at their disposal to solve real-world problems.

<u>Suggested change:</u> For SMP.4, add the numerical derivation of the 3 coefficients of the scaled PS(t) function as an example of homework exercise, provide appropriate outline of the methodology as hints.

------------(End comment 4)--

Issue 5): Equation of a line*. *Appendix A, *oddly the same information/description of 'equation of a line' appears verbatim twice in the framework as described below.*

1st appearance: *pp 15, portion of Figure/Table of "Standards for Mathematical Practice examples for students, … , practice in Algebra I", embedded between lines 89 and 90.*

MP8. Look for and express regularity in repeating reasoning.	*Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression or points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, **the equation** **will give a general equation of that line.***

2nd appearance: *pp 42, portion of Figure A.11, embedded within lines 492-493. Figure A.11: Standards for Mathematical Practice—Explanation and Examples for Mathematics*

SMP.8 *Look for and express regularity in repeated reasoning.*	*Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, **the equation** **will give a general equation of that line**.*

------------(Begin comment 5)--

There is a typo, on pp 15, "MP8. … *in repeating reasoning" should read "MP8…in **repeated** reasoning."*

Regarding the line equation , although mathematically correct, nonetheless the way it is written might be confused with that of the equation for the slope of a line, and there are infinitely many lines with the same slope. One would rather take care to aid students in the understanding of the concept of a line. As noted, the same definition for the equation of a line appears identically in two distinct places in the second field review draft.

Suggested change: i) Rewrite equation of line as $(y-y_1) = m(x-x_1)$ or $y = mx + (y_1 - mx_1)$ which is more standard and consistent with the earlier framework definition of a line $y = mx + c$, and

ii) framework authors decide where to place the definition for the equation for a line (say pp 15, MP8 on its 1st appearance), and replace its 2nd redundant appearance on pp. 42 with an appropriate definition of SMP.8.

------------(End comment 5)--

Issue 6): Recursive Sequences. *Revised Math Framework, Appendix A, pp 34, portion of Figure A.8 embedded within lines 409 -410.*

Figure A.8: High School Integrated 1 Content Connections, Big Ideas, and Standard

Taking Wholes Apart, Putting Parts Together	Composing Functions	***F-BF.3, F-BF.2, F-IF.3:*** *Build and explore new functions that are made from existing functions, and explore graphs of the related functions using technology****. Recognize sequences are functions and are defined recursively.***

------------(Begin comment 6)--**Comment 7):** Framework definition specifies that "Recognize sequences are functions and are defined "recursively", which is evidentially false as non-recursive sequences exist, and plentiful. A non-recursive sequence has a *non-recursive formula* that does not itself depend on any other terms in the sequence. For example,

$S_n = n^2$

is one of the most basic **non-recursive** sequence formulas. S_n represents the nth number in the sequence, and its formula is n^2, which can be directly computed by plugging in the value of n you want. The restriction of discussion to only recursive sequences imparts an incomplete and erroneous characterization of the subject.

Suggested change: Modify the definition by inserting the word '*OFTEN*' to the statement "… Recognize sequences are functions and are *OFTEN* defined recursively."

------------(End comment 6)--

------------(End Write-Up)---